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THE CONVERGENCE OF THE METHOD OF GENERALIZED REACTION IN CONTACT PROBLEMS WITH A FREE BOUNDARY†

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The convergence of the method of generalized reaction proposed in [1] to solve contact problems with an unknown region of active interaction (with a free boundary) is proved with fairly general initial assumptions. An example of the combined cylindrical bending of two rectangular plates is given.

1. WE WILL assume that the displacements u_1 and u_2 of two elastic components of a structure (rods, plates or shells) under certain loads lead to contact interaction between two components. We will assume that the deformation of each of these components is described by stationary linear equations and the functions u_1 and u_2, respectively

$$(L_i u_i)(P) = f_i(P), \quad P \in \Omega_i \subset R^{n_i}$$
$$(\Gamma_{i,j} u_i)(P) = 0, \quad P \in \partial\Omega_i, \quad j \in 1: r_i, \quad i = 1, 2 \tag{1.1}$$

Certain operators in $L_2(\Omega_i)$ $(i = 1, 2)$ or other Hilbert spaces correspond to boundary-value problems (1.1). The regions in which these operators are defined are linear functions, which are continuously differentiable a sufficient number of times and satisfy boundary conditions (1.1). Hence, the solution of problems (1.1) reduce to the solution of the operator equations

$$A_i u_i = f_i(P), \quad P \in \Omega_i, \quad i = 1, 2 \tag{1.2}$$

In most cases (and we will assume this below) the boundary-value problems are self-conjugate in Lagrange's sense, while the operators are positive definite in dense sets of corresponding Hilbert spaces. The operators A_i can then be extended to self-conjugate operators. We will assume that this extension is carried out.

Suppose the gap between thin-walled components in the region of possible contact Ω_0 $(\Omega_0 \subset \Omega_1 \cap \Omega_2)$ is defined by the function $\Delta(P)$, $P \in \Omega_0$. Then the contact problem considered can be formulated as follows:

$$A_1 u_1 = f_1(P) - x(P) H(P), \quad P \in \Omega_1$$
$$A_2 u_2 = f_2(P) + x(P) H(P), \quad P \in \Omega_2 \tag{1.3}$$

$$x(P) \geqslant 0, \quad P \in \Omega_0$$
$$u_1(P) - u_2(P) - \Delta(P) \leqslant 0, \quad P \in \Omega_0 \tag{1.4}$$
$$x(P)[u_1(P) - u_2(P) - \Delta(P)] = 0, \quad P \in \Omega_0$$

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The first relation (1.4) is the condition that the coupling should be one-sided, the second is the condition for non-penetration, and the third is the condition of supplementing flexibility: if $x>0$, we have $u_1-u_2-\Delta=0$ and if $u_1-u_2-\Delta<0$, we have $x=0$; $H(P)=1$ when $P\in\Omega_0$, and $H(P)=0$ when $P\notin\Omega_0$.

From Eqs (1.3) we have

$$x(P)=\tfrac{1}{2}(A_2u_2-A_1u_1+f_1-f_2)(P),\quad P\notin\Omega_0 \tag{1.5}$$

Conditions (1.4) will be satisfied if $x(P)$ satisfies the equation

$$x(P)=[x-\alpha(u_2-u_1+\Delta)]_+(P),\quad \alpha>0,\quad P\in\Omega_0 \tag{1.6}$$

(the subscript plus indicates the positive part of the corresponding function $\varphi_+=\tfrac{1}{2}(\varphi+|\varphi|)$).

In fact, we see directly from (1.6) that the first condition of (1.4) is satisfied. We will further assume that the expression in the square brackets is negative. Then these brackets can be omitted, and we therefore obtain the condition

$$u_1-u_2-\Delta=0 \tag{1.7}$$

If the expression in the square brackets in (1.6) is negative

$$x-\alpha(u_2-u_1+\Delta)<0 \tag{1.8}$$

we have

$$x=0 \tag{1.9}$$

and we obtain the following condition from (1.8)

$$u_1-u_2-\Delta<0 \tag{1.10}$$

It is obvious that the relations (1.7)–(1.10) ensure that conditions (1.4) are satisfied.

Hence, we can consider the following system instead of (1.3) and (1.4)

$$A_1u_1=f_1-x,\quad A_2u_2=f_2+x,\quad x=[x-\alpha(u_2-u_1+\Delta)]_+ \tag{1.11}$$

The method of solving contact problems in mechanics with an unknown region of active interaction $\Omega_0^*\subset\Omega_0$ based on the use of (1.6) was called the method of generalized reaction in [1].

If we assume that inverse operators A_i^{-1} exist and are defined, then using (1.11) we obtain the following equation for the contact reaction

$$x=[x-\alpha\Phi'(x)]_+$$
$$(\Phi'\triangleq A_2^{-1}(f_2+x)-A_1^{-1}(f_1-x)+\Delta) \tag{1.12}$$

We will assume that A_i are positive definite operators specified in a unit real Hilbert space $L_2(\Omega)$ $(\Omega=\Omega_1=\Omega_2=\Omega_0)$, i.e.

$$\langle A_iu,u\rangle\geqslant\gamma_i^2\|u\|^2,\quad i=1,2 \tag{1.13}$$

We recall that these conditions guarantee that the inverse operators A_1^{-1} and A_2^{-1} are self-conjugate, where

$$\| A_i^{-1} \| \leqslant 1/\gamma_i^2 \quad i = 1, 2 \tag{1.14}$$

We will set up the functional with respect to $\Phi'(x)$. We have

$$\Phi(x) = ½ \langle A_1^{-1}(f_1 - x), f_1 - x \rangle + ½ \langle A_2^{-1}(f_2 + x), f_2 + x \rangle + \langle x, \Delta \rangle \tag{1.15}$$

It is obvious that $\Phi(x) \geqslant 0$, if $x \in M\{x \geqslant 0\}$, and, consequently, the following exists

$$\inf_{x \in M} \Phi(x) = \Phi_*$$

The functional $\Phi(x)$ is strictly convex.

In fact, suppose $x' \neq x''$, while u_i', u_i'' are solutions of Eqs (1.3) corresponding to these reactions, i.e. $u_i' \neq u_i''$ $(i = 1, 2)$. Then

$$\begin{aligned} &\Phi\left(\frac{x' + x''}{2}\right) - \frac{1}{2}\Phi(x') - \frac{1}{2}\Phi(x'') = \\ &= -1/8 \langle A_1 (u_1'' - u_1'), u_1'' - u_1' \rangle - 1/8 \langle A_2 (u_2'' - u_2'), u_2'' - u_2' \rangle < 0 \end{aligned} \tag{1.16}$$

The functional $\Phi(x)$ can be written in the form

$$\begin{aligned} &\Phi(x) = ½ \langle Gx, x \rangle + \langle q, x \rangle + \varphi_0 \\ &G = A_1^{-1} + A_2^{-1}, \quad g = A_2^{-1} f_2 - A_1^{-1} f_1 + \Delta \\ &\varphi_0 = ½ \langle A_1^{-1} f_1, f_1 \rangle + ½ \langle A_2^{-1} f_2, f_2 \rangle \end{aligned}$$

Taking inequalities (1.14) into account we obtain

$$\begin{aligned} &\langle Gx, x \rangle \leqslant \| G \| \| x \|^2 \leqslant \mu_0 \quad \|x\|^2 \\ &(\mu_0 = (\gamma_1^2 + \gamma_2^2)/(\gamma_1^2 \gamma_2^2)) \end{aligned}$$

On the basis of (1.12) we can form the following iterative scheme

$$x_{n+1} = [x_n - \alpha \Phi'(x_n)]_+ \triangleq \omega(x_n) \tag{1.17}$$

It can be shown that $\{x_n\}$ is a minimizing sequence for $\Phi(x)$.

To do this we will first obtain the inequality [2]

$$\langle \Phi'(x_n), x_{n+1} - x_n \rangle \leqslant -\alpha^{-1} \| x_{n+1} - x_n \|^2 \tag{1.18}$$

Consider the auxiliary functional

$$\Psi(z) = ½ \| z - (x_n - \alpha \Phi'(x_n)) \|^2, \quad z \in M$$

It is obvious that

$$\arg\min_{z \in M} \Psi(z) = [x_n - \alpha \Phi'(x_n)]_+ = x_{n+1}$$

Hence, the necessary and sufficient condition for a minimum of $\Psi(x)$ on the element x_{n+1} can be written in the form of the inequality

$$\langle \Psi'(x_{n+1}),\ z - x_{n+1} \rangle \geqslant 0, \quad \forall z \in M \tag{1.19}$$

Assuming here that $z = x_n$ and taking into account the fact that

$$\Psi'(x_{n+1}) = x_{n+1} - x_n + \alpha \Phi'(x_n)$$

we arrive at inequality (1.18).

Using the formula of finite increments, we can write

$$\begin{aligned} \Phi(x_{n+1}) &= \Phi(x_n) + \langle \Phi'(x_n),\ x_{n+1} - x_n \rangle + \\ &+ \theta \langle G(x_{n+1} - x_n),\ x_{n+1} - x_n \rangle, \quad \theta \in (0, 1) \end{aligned} \tag{1.20}$$

Hence, taking into account the limits (1.16) and (1.18) we obtain

$$\Phi(x_{n+1}) \leqslant \Phi(x_n) + (\mu_0 - \alpha^{-1}) \| x_{n+1} - x_n \|^2. \tag{1.21}$$

It can be seen from (1.21) that for fairly small α ($\alpha < 1/\mu_0$) the sequence $\{\Phi(x_n)\}$ decreases, but $\Phi(x) \geqslant 0$, and hence as $n \to \infty$

$$\begin{aligned} &\| x_{n+1} - x_n \| \to 0, \quad \Phi(x_n) \to \Phi_x \geqslant \Phi_* \\ &\langle G(x_{n+1} - x_n),\ x_{n+1} - x_n \rangle \to 0 \end{aligned} \tag{1.22}$$

(Note that the convergence of the sequence $\{x_n\}$ in $L_2(\Omega)$ does not follow from conditions (1.22).)

Finally, from (1.20) we have

$$\lim_{n \to \infty} \langle \Phi'(x_n),\ x_{n+1} - x_n \rangle = 0 \tag{1.23}$$

Now consider, together with $\{x_n\}$ the sequence $\{y_n\}$, such that

$$y_{n+1} = [y_n - \alpha \Phi(y_n)]_+, \quad \Phi(y_0) \leqslant \Phi_* + \epsilon \tag{1.24}$$

where ϵ is a positive number, as small as desired.

We will show that the distance between the elements x_n and y_n is uniformly bounded by a certain number R_0. We have

$$\begin{aligned} \| x_{n+1} - y_{n+1} \|^2 &= \| \omega(x_n) - x_n - \omega(y_n) + y_n + x_n - y_n \|^2 = \\ &= \| \omega(x_n) - x_n - \omega(y_n) + y_n \|^2 + \| x_n - y_n \|^2 \\ &+ 2 \langle \omega(x_n) - x_n - \omega(y_n) + y_n,\ x_n - y_n \rangle \end{aligned} \tag{1.25}$$

Since the distance between the projection of the elements on the convex set (M) does not exceed the distance between the projected elements, we have

$$\| \omega(x_n) - \omega(y_n) \| \leqslant \| x_n - \alpha \Phi'(x_n) - y_n + \alpha \Phi'(y_n) \|$$

Hence we obtain (I is the identity operator)

$$\langle \omega(x_n) - \omega(y_n),\ x_n - y_n \rangle \leqslant \| I - \alpha G \| \| x_n - y_n \|^2$$

and, consequently, for fairly small α

$$\begin{aligned} &\langle \omega(x_n) - x_n - \omega(y_n) + y_n,\ x_n - y_n \rangle \leqslant \\ &\leqslant (\| I - \alpha G \| - 1) \| x_n - y_n \|^2 \leqslant 0. \end{aligned}$$

Hence, in view of (1.25) we arrive at the inequality

$$\| x_{n+1}-y_{n+1} \|^2 \leqslant 2 \| x_{n+1}-x_n \|^2 + $$
$$+ 2 \| y_{n+1}-y_n \|^2 + \| x_n-y_n \|^2$$

Hence we have

$$\| x_{n+1}-y_{n+1} \|^2 \leqslant \sum_{k=0}^{n} \| x_{k+1}-x_k \|^2 +$$
$$+ 2 \sum_{k=0}^{n} \| y_{k+1}-y_k \|^2 + \| x_0-y_0 \|^2 \leqslant R_0^2 \tag{1.26}$$
$$R_0^2 = \frac{2\,\alpha_0}{1-\alpha_0\,\mu_0} (\Phi(x_0) + \Phi(y_0) - 2\,\Phi_*) + \| x_0-y_0 \|^2$$

Here we have borne in mind that (see (1.21))

$$\sum_{k=0}^{n} \| x_{k+1}-x_k \|^2 \leqslant \frac{\alpha_0}{1-\alpha_0\,\mu_0} (\Phi(x_0)-\Phi_*)$$
$$\sum_{k=0}^{n} \| y_{k+1}-y_k \|^2 \leqslant \frac{\alpha_0}{1-\alpha_0\mu_0} (\Phi(y_0)-\Phi_*)$$
$$(\alpha_0 < 1/\mu_0)$$

We will now consider the variational inequality (1.19). Assuming that $z = y_n$ in it we obtain after elementary reduction

$$\alpha \langle \Phi'(x_n), y_n-x_n \rangle \geqslant \alpha \langle \Phi'(x_n), x_{n+1}-x_n \rangle +$$
$$+ \langle x_{n+1}-x_n, x_n-y_n \rangle + \| x_{n+1}-x_n \|^2$$

Taking relations (1.22), (1.23) and (1.26) into account, it can be shown that

$$\lim_{n\to\infty} \langle \Phi'(x_n), y_n-x_n \rangle \geqslant 0 \tag{1.27}$$

In view of the convexity of the functional $\Phi(x)$ we have

$$\Phi(y_n) \geqslant \Phi(x_n) + \langle \Phi'(x_n), y_n-x_n \rangle$$

Passing to the limit in this inequality and taking (1.27) into account we obtain

$$\lim_{n\to\infty} \Phi(y_n) \triangleq \Phi_y \geqslant \Phi_x$$

We finally obtain

$$\Phi_* \leqslant \Phi_x \leqslant \Phi_y \leqslant \Phi(y_0) \leqslant \Phi_* + \epsilon$$

i.e. $\{x_n\}$ is a minimizing sequence for the functional $\Phi(x)$ for any $x_0 \in M$.

2. The contact problem considered in Sec. 1 allows of the following energy formulation

$$J(u_1, u_2) \to \min_{u_1-u_2-\Delta \leqslant 0}$$
$$J(u_1, u_2) = \tfrac{1}{2}\langle A_1u_1, u_1 \rangle + \tfrac{1}{2}\langle A_2u_2, u_2 \rangle - \tag{2.1}$$
$$- \langle f_1, u_1 \rangle - \langle f_2, u_2 \rangle$$

Using the sequence of reactions generated by scheme (1.17) we can construct series for the displacements from the formulae

$$u_1^{(n)} = A_1^{-1}(f_1 - x_n), \quad u_2^{(n)} = A_2^{-1}(f_2 + x_n) \tag{2.2}$$

We will show that the series $\{u_i^{(n)}\}$ $(i = 1, 2)$ converge to the solution of problem (2.1).
First of all we have (see (1.16))

$$\Phi\left(\frac{x_n + x_m}{2}\right) - \frac{1}{2}\Phi(x_n) - \frac{1}{2}\Phi(x_m) \leqslant$$
$$\leqslant -{}^1\!/_8\, \gamma_1^2 \, \| u_1^{(n)} - u_1^{(m)} \|^2 - {}^1\!/_8\, \gamma_2^2 \, \| u_2^{(n)} - u_2^{(m)} \|^2 \tag{2.3}$$

Hence it follows that the following exist

$$\lim_{n \to \infty} u_i^{(n)} \triangleq u_i^*, \quad i = 1, 2 \tag{2.4}$$

Introducing the Lagrangian

$$\Lambda(u_1, u_2, x) = J(u_1, u_2) + \langle x, u_1 - u_2 - \Delta \rangle, \quad x \in M,$$

we can reformulate problem (2.1) as follows:

$$\sup_{x \in M} \Lambda(u_1, u_2, x) \to \min_{u_1, u_2} \tag{2.5}$$

Suppose $\{x_n\}$ is a minimizing sequence for $\Phi(x)$ while $\{M_n\}$ is a sequence of weakly compact sets

$$M_n = \{ x \in L_2(\Omega) : 0 \leqslant x \leqslant d_n \} \tag{2.6}$$
$$d_n \to \infty \quad \text{as} \quad n \to \infty$$

such that $x_n \in M_n$. Consider the auxiliary problem

$$\max_{x \in M_n} \Lambda(u_1, u_2, x) \to \min_{u_1, u_2} \tag{2.7}$$

Taking into account the fact that the functional $\Lambda(u_1, u_2, x)$ is convex in u_1 and u_2, and linear in x, and that M_n is a weakly compact set, we can change from (2.7) to the dual problem [3]

$$\min_{u_1, u_2} \Lambda(u_1, u_2, x) \to \max_{x \in M_n} \tag{2.8}$$

The minimization problem in (2.8) for fixed x is equivalent to solving the two equations in (1.11). In this case

$$\Lambda(u_1, u_2, x) = -\Phi(x)$$

and problem (2.8) takes the form

$$\Phi(x) \to \min_{x \in M_n}$$

Suppose

$$\arg\min_{x \in M_n} \Phi(x) = x_n^*$$

Then, taking into account the conditions for constructing the series $\{M_n\}$ we have

$$\Phi(x_n) \geq \Phi(x_n^*) \geq \Phi_*$$

i.e. $\{x_n^*\}$ is a minimizing sequence for $\Phi(x)$.

Further, introducing the notation

$$u_{1,n} \triangleq A_1^{-1}(f_1 - x_n^*), \quad u_{2,n} \triangleq A_2^{-1}(f_2 + x_n^*) \tag{2.9}$$

and taking into account the fact that the point $(u_{1,n}, u_{2,n}, u_n^*)$ is a saddle point for the functional $\Lambda(u_1, u_2, x)$ in the set $(L_2 \times L_2 \times M_n)(\Omega)$, we can write

$$\Lambda(u_{1,n}, u_{2,n}, x) \leq \Lambda(u_{1,n}, u_{2,n}, x_n^*) \leq \Lambda(u_1, u_2, x_n^*) \tag{2.10}$$

We will first show that

$$\lim_{n\to\infty} u_{i,n} = \lim_{n\to\infty} u_i^{(n)} = u_i^* \tag{2.11}$$

Consider the combined sequence

$$\{\tilde{x}_n\} \triangleq \{x_n, x_n^*\}$$

It is obvious that $\Phi(\tilde{x}_n) \underset{n\to\infty}{\rightarrow} \Phi_*$ and, consequently, the following limits exist

$$\lim_{n\to\infty} \tilde{u}_{i,n} = \tilde{u}_i^*, \quad i = 1, 2$$

$$\tilde{u}_{1,n} = A_1^{-1}(f_1 - \tilde{x}_n), \quad \tilde{u}_{2,n} = A_2^{-1}(f_2 + \tilde{x}_n)$$

Then

$$\lim_{n\to\infty} u_{i,n} = \lim_{n\to\infty} u_i^{(n)} = \tilde{u}_i^*$$

which proves (2.11).

We will further show that the functions u_i^* $(i = 1, 2)$ satisfy the limitations of problem (2.1). Let us assume the contrary. Suppose a set of non-zero measure $\Omega' \subset \Omega$ exists, such that

$$u_1^* - u_2^* - \Delta > 0 \text{ on } \Omega' \tag{2.12}$$

We will introduce the notation

$$K \triangleq \int_{\Omega'} (u_1^* - u_2^* - \Delta)\, d\Omega > 0 \tag{2.13}$$

Then, by (2.4) the following inequality is satisfied for fairly large values of n

$$K_n = \int_{\Omega'} (u_1^{(n)} - u_2^{(n)} - \Delta)\, d\Omega > \frac{1}{2} K \tag{2.14}$$

Further, using the left-hand inequality of (2.10) and taking the second relation of (2.1) into account, we have

$$\Lambda(u_{1n}, u_{2n}, x_n^*) \geqslant -\langle f_1, u_{1,n}\rangle - \langle f_2, u_{2,n}\rangle + \int_\Omega x(u_{1,n}-u_{2,n}-\Delta)\, d\Omega$$

The last inequality holds for any $x \in M_n$. Assuming that

$$x(P) = \begin{cases} d_n, & P \in \Omega' \\ 0, & P \notin \Omega' \end{cases}$$

we obtain

$$\begin{aligned} &\Lambda(u_{1,n}, u_{2,n}, x_n^*) = -\Phi(x_n^*) \geqslant \\ &\geqslant c_n + \tfrac{1}{2} d_n K \underset{n\to\infty}{\to} \infty \\ &c_n \underset{n\to\infty}{\to} -\langle f_1, u_1^*\rangle - \langle f_2, u_2^*\rangle > -\infty \end{aligned} \tag{2.15}$$

It is obvious that relation (2.15) contradicts the condition $\Phi(x) \geqslant 0$ and, consequently, assumption (2.12) is untenable, i.e.

$$u_1^* = u_2^* - \Delta \leqslant 0 \text{ almost everywhere on } \Omega \tag{2.16}$$

We will show that

$$\lim_{n\to\infty} \langle x_n^*, u_{1,n} - u_{2,n} - \Delta\rangle = 0 \tag{2.17}$$

We have from the left-hand side of inequality (2.10)

$$\begin{aligned} &\langle x, u_{1,n} - u_{2,n} - \Delta\rangle \leqslant \langle x_n^*, u_{1,n} - u_{2,n} - \Delta\rangle \\ &\forall x \in M_n \end{aligned}$$

Hence it follows that

$$\langle x_n^*, u_{1,n} - u_{2,n} - \Delta\rangle \geqslant 0 \tag{2.18}$$

since, otherwise, when $x = 0$ we obtain a contradiction. We will assume that a sequence n_k exists such that

$$\langle x_{n_k}^*, u_{1,n_k} - u_{2,n_k} - \Delta\rangle \geqslant a_0 > 0$$

Using the right-hand inequality in (2.10) and assuming in it that $u_1 = u_1^*$, $u_2 = u_2^*$, taking (2.16) into account we obtain

$$J(u_{1,n_k}, u_{2,n_k}) + a_0 \leqslant J(u_1^*, u_2^*)$$

Passing to the limit here we obtain a contradiction with the fact that $a_0 > 0$.

Suppose $\bar{u}_i$ $(i = 1, 2)$ are geometrically admissible displacements, i.e. such that, in particular, the condition $\bar{u}_1 - \bar{u}_2 = \Delta \leqslant 0$ is satisfied for them. From the right-hand inequality of (2.10) we have

$$\begin{aligned} &J(u_{1,n}, u_{2,n}) + \langle x_n^*, u_{1,n} - u_{2,n} - \Delta\rangle \leqslant \\ &\leqslant J(\bar{u}_1, \bar{u}_2) + \langle x_n^*, \bar{u}_1 - \bar{u}_2 - \Delta\rangle \leqslant J(\bar{u}_1, \bar{u}_2) \end{aligned}$$

Passing to the limit here and taking (2.11) and (2.17) into account we obtain

$$J(u_1^*, u_2^*) \leqslant J(\bar{u}_1, \bar{u}_2)$$

for any $\bar{u}_i \in L_2(\Omega)$ which satisfy the second inequality of (1.4)

Note 1. The iterative scheme (1.17) can be obtained if we formally (not using the weakly compact sets M_n) replace (2.5) by the dual problem

$$\min_{u_1, u_2} \Lambda(u_1, u_2, x) \to \sup_{x \in M}$$

transfer from it to the minimization problem

$$\Phi(x) \to \inf_{x \in M}$$

and apply the method of gradient projection to this problem. In this case, the Lagrange multiplier can be interpreted as a contact reaction. The scalar parameter α, which regularizes the step of gradient descent, has no effect on the values of u_1 and u_2, determined using the iterative scheme

$$A_1 u_1^{(n+1)} = f_1 - \tfrac{1}{2}[A_2 u_2^{(n)} - A_1 u_1^{(n)} - f_2 + f_1 - \alpha(u_2^{(n)} - u_1^{(n)} + \Delta)]_+$$

$$A_2 u_2^{(n+1)} = f_2 + \tfrac{1}{2}[A_2 u_2^{(n)} - A_1 u_1^{(n)} - f_2 + f_1 - \alpha(u_2^{(n)} - u_1^{(n)} + \Delta)]_+$$

3. We will consider the contact problem for two parallel rectangular plates clamped and loaded so that the shape of their deflection is cylindrical. To fix our ideas we will assume that the hinge-support conditions are satisfied at two opposite edges of the plate (the distance between which is l). We will also assume that the upper plate is acted upon by a normal load $q(\xi)$, while the lower plate experiences only a pressure $x(\xi)$ from the side of the upper plate, when the bowing of the latter exceeds the value of the initial gap between the plates $\Delta = \text{const}$. This contact problem can be formulated as follows:

$$\begin{gathered} d_0 w_1^{IV} = q(\xi) - x(\xi), \quad \xi \in (0, l) \\ w_1(0, \eta) = w_1(l, \eta) = w''_{1,\xi\xi}(0, \eta) = w''_{1,\xi\xi}(l, \eta) = 0 \end{gathered} \tag{3.1}$$

$$\begin{gathered} d_0 w_2^{IV} = x(\xi), \quad \xi \in (0, l) \\ w_2(0, \eta) = w_2(l, \eta) = w''_{2,\xi\xi}(0, \eta) = w''_{2,\xi\xi}(l, \eta) = 0 \end{gathered} \tag{3.2}$$

$$x(\xi) \geqslant 0, \quad \xi \in (0, l) \tag{3.3}$$

$$w_1(\xi, \eta) \leqslant w_2(\xi, \eta) + \Delta \tag{3.4}$$

$$x(\xi)[w_1(\xi, \eta) - w_2(\xi, \eta) - \Delta] = 0 \tag{3.5}$$

(d_0 is the cylindrical stiffness of the plate). Green's function for boundary-value problems (3.1) and (3.2) has the form

$$\begin{gathered} G(\xi, t) = A(\xi - t)_+^3 + \varphi(t)\xi^3 + \psi(t)\xi \\ A = \frac{1}{6d_0}, \quad \varphi(t) = -\frac{l-t}{6d_0 l}, \quad \psi(t) = \frac{t(l-t)(2l-t)}{6d_0 l} \end{gathered} \tag{3.6}$$

We will use the method of splitting the initial boundary-value problem into simpler problems, assuming that the action of the upper plate on the lower plate is described by a function of the form

$$\begin{gathered} x(\xi) = R_1\delta(\xi - \xi_1) + R_2\delta(\xi - (l - \xi_2)) + x_0(\xi) \\ \xi \in [\xi_1, l - \xi_2] \end{gathered} \tag{3.7}$$

($\delta(\cdot)$ is the delta function). Assuming that

$$w_1(\xi,\eta)=w_2(\xi,\eta),\quad \xi\in[\xi_1, l-\xi_2] \tag{3.8}$$

we obtain

$$x_0(\xi)=\tfrac{1}{2}q(\xi)\,[H(\xi-\xi_1)-H(\xi-(l-\xi_2))] \tag{3.9}$$

($H(\cdot)$ is Heaviside's function). Using relations (3.6)–(3.9) the bowings of the plates are given by the equations

$$w_1(\xi,\eta)=\int_0^l G(\xi,t)\,q(t)\,dt-\int_{\xi_1}^{l-\xi_2} G(\xi,t)\,q(t)\,dt-$$
$$-R_1G'(\xi,\xi_1)-R_2G(\xi,l-\xi_2) \tag{3.10}$$
$$w_2(\xi,\eta)=\frac{1}{2}\int_{\xi_1}^{l-\xi_2} G(\xi,\xi_1-t)\,q(t)\,dt+R_1G'(\xi,\xi_1)+R_2G(\xi,l-\xi_2)$$

Using (3.8) and (3.10) we have

$$\int_0^{\xi_1} tq(t)\,dt=2R_1\xi_1,\qquad \int_0^{\xi_2} tq(l-t)\,dt=2R_2\xi_2 \tag{3.11}$$

$$\int_0^{\xi_1}(\xi_i^2-t^2)\,tq(t)=6d_0\Delta,\qquad i=1,2 \tag{3.12}$$

By analysing (3.12) it can be shown that each of them can have only one solution, i.e. the contact region is always simply connected. Further, an interesting property of the boundary-value problem considered emerges from Eqs (3.7): if as a result of a certain loading of the upper plate (independent of η) one obtains a contact region, the position and extent of this region with respect to ξ, and also the values of the concentrated reactions, are independent of the load within the contact zone.

Hence, the solution of the contact problem reduces to realizing an extremely simple algorithm: (1) obtain the roots ξ_1 and ξ_2 of Eqs (2.13), (2) determine the concentrated reactions R_1 and R_2 from Eqs (3.11), and (3) calculate the functions w_1 and w_2 from (3.10).

Consider the special case when $q(\xi)=q_0=\text{const}$. From (3.12) and (3.11) successively we obtain

$$\xi_1=\xi_2=\sqrt{24\,\Delta d_0/q_0}\triangleq\xi_0,\qquad R_1=R_2={}^1/_4\,q_0\,\xi_0 \tag{3.13}$$

In this case, the condition for obtaining a contact zone has the form

$$q_0>384\,\Delta d_0/l^4$$

Taking (3.13) into account we obtain from (3.10)

$$w_i(\xi,\eta)=\frac{q_0}{24d_0}\times\begin{cases}U_i(\xi), & \xi\leqslant\xi_0\\ V_i(\xi), & \xi_0\leqslant\xi\leqslant\tfrac{1}{2}l\end{cases} \tag{3.14}$$

$$U_1=\xi^4-(l+\xi_0)\,\xi^3+(\tfrac{1}{2}l^3+\xi_0^3)\,\xi$$

$$U_2=\xi_0^4-(l-\xi_0)\,\xi^3+(\tfrac{1}{2}l^3-\xi_0^3)\,\xi$$

$$V_1=V_2=\tfrac{1}{2}\xi^4-l\,\xi^3+\tfrac{1}{2}l^3\xi+\tfrac{1}{2}\xi_0^4$$

Note 2. The example considered above is sufficient to show that the series $\{x_n\}$ (see (1.17)), generally speaking, does not converge in $L_2(\Omega)$. However, as was shown in Sec. 2, the sequences $\{u_i^{(n)}\}$ (see (2.2)), obtained using $\{x_n\}$, converges to the solution of the contact problem on average. The operator form of the description does not enable us here to show that, in specific problems, the sequences $\{u_i^{(n)}\}$ converge to the solution with a certain number of its derivatives. For example, for the system of two cylindrical curved plates considered in Sec. 3, we have the relation (see (1.16))

$$\Phi\left(\frac{x_n+x_m}{2}\right)-\frac{1}{2}\Phi(x_n)-\frac{1}{2}\Phi(x_m)=$$
$$=-\frac{1}{8}d_0 a\left\{\left\|\frac{d^2}{d\xi^2}(w_1^{(n)}-w_1^{(m)})\right\|^2+\left\|\frac{d^2}{d\xi^2}(w_2^{(n)}-w_2^{(m)})\right\|^2\right\}$$

(a is the width of the plate). Hence it follows that the sequences $\{d^2w_i^{(n)}/d\xi^2\}$ converge on average, while $\{w_i^{(n)}\}$, $\{dw_i^{(n)}/d\xi\}$ converges uniformly with respect to ξ.

REFERENCES

1. MIKHAILOVSKII Ye. I. and TARASOV V. N., Contact problems for flexible structural components. In *Problems of the Non-Linear Theory of Elasticity*, Izd. Kalinin. Politekh. Univ., Kalinin, 1989.
2. DEM'YANOV V. F. and RUBINOV A. M., *Approximate Methods of Solving Extremal Problems.* Izd. Leningrad Univ., Leningrad, 1980.
3. BALAKRISHNAN A. V., *Applied Functional Analysis.* Nauka, Moscow, 1980.

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